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Quantum dynamical mappings and Markovianity

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Abstract

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The description of the evolution of open quantum systems is a topic of great interest both for its wide range of technological applications and for the understanding of fundamental aspects of quantum theory. In this context, the quantum dynamical mappings formalism is a widely used tool for generally characterizing open systems dynamics. Nonetheless, for this formalism there is no agreement on how to define Markovianity, which is an extremely important concept for most informational analysis of quantum systems. In this undergraduate thesis we compare the quantum dynamical mappings framework with the stochastic processes theory, which has a precise definition of Markovianity, and show that there is a fundamental difference between them that prevents us from properly defining Markovianity in the quantum case. In the end, we briefly present an alternative approach, the process tensor framework, in which is possible to consistently define quantum Markovianity. We then argue that, for this reason, the process tensor framework might be more adequate than quantum dynamical mappings for generally describing the evolution of open quantum systems.

Keywords: Quantum dynamical mappings. Quantum operations. Markovianity. Divisibility.

Contents

1 Introduction

Quantum theory is one of the most successful physical theories. For the last one hundred years it has been proving itself to be the correct way to describe a wide variety of physical systems, if not all of them. Despite being such a well consolidated theory, there are still many ongoing debates on some aspects of its underlying formal structure. The one we focus here is on how to describe the evolution of *open quantum systems*.

Since John von Neumann's 1932 seminal work, the mathematical structure of quantum theory remains nearly untouched. (1) It is based on four axioms that could be listed as: description and composition of quantum systems, evolution of closed systems and measurement. Notice that the most general description of the evolution of quantum systems should also address open systems, but this is not made explicit on the axioms that define quantum theory.

The first serious approach to this problem was made by Jordan and Sudarshan in their 1961 (2) and 1962 papers. (3) The authors proposed that, in general, the evolution of quantum systems should be simply the one given by operators that preserve the properties of quantum states. By considering a family of these operators that map the initial state of the system to some later state, we could obtain a complete description of the physical process. This approach to the evolution of quantum systems became known as *quantum dynamical mappings* (or also *quantum dynamical maps* or *quantum operations*).

In the subsequent decades it turned out that the quantum dynamical mappings framework was a powerful tool for generally describe the dynamics of open quantum systems. Much of the progress on quantum theory made then was entirely based on this approach, specially for quantum information related fields like quantum data processing (4), quantum error correction (5), quantum computation (6) and quantum process tomography. (7)

In the first years of the twenty-first century, however, a new problem was raised by quantum information physicists. Mainly motivated by the common objective of building a quantum computer, the studies of quantum information storing and processing converged towards a fundamental question: Which quantum processes keep information from the past and which do not? An answer to this question would be useful as systems that keep information could be used for storing it, while systems that do not keep information are usually easier to describe, so they could be used to model different types of environments.

In order to characterize which quantum processes do not have memory of their past states, Rivas, Huelga and Plenio pointed out that there were a lot of similarities between quantum dynamical mappings and the stochastic processes formalism, so that it would be interesting to borrow the notion of Markovianity from stochastic processes to quantum theory. (8) They realized that a direct extension of the definition would not be possible for quantum dynamical mappings as, despite the similarities, there was a difference between the two formalisms related to the measurement process.

In this context, many non-equivalent proposals of definitions for Markovianity of quantum dynamical mappings were made, like the monotonicity of some information quantifier (9), the divisibility of the dynamics (10), the detection of initial correlations (11), changes to quantum coherence (12), channels capacity (13) and positivity of quantum maps. (14) Each one of these definitions resembles some aspect of the original definition for stochastic processes, but there is no agreement on which definition should be taken as the correct one.

More recently, it has been shown that the problem behind defining quantum Markovianity lies on the structure of quantum dynamical mappings formalism: its way of describing the dynamics does not have the necessary element for such a definition. (15) While Markovianity is a property on the multi-time correlations of a process, a quantum dynamical mapping provides only a set of two-time correlations.

The purpose of this undergraduate thesis is to discuss the aforementioned aspects of the definition of quantum Markovianity. It is structured as follows: in Chapter 2 we present the stochastic processes theory; in Chapter 3 we formalize quantum dynamical mappings by establishing the relations between quantum theory and stochastic processes theory; in Chapter 4 we discuss some of the proposed definitions of quantum Markovianity and argue why it is not possible to properly define Markovianity for quantum dynamical mappings; finally, in Chapter 5 we present some conclusions and final remarks.

2 Stochastic Processes

We begin this chapter by briefly introducing the basic concepts of probability theory, which then allow us to define stochastic processes. The main objective here is to present and discuss the properties of divisibility and Markovianity and the relation between them. We also show some simple examples of stochastic processes to clarify how these properties are related with the approach we use to describe the process.

Probability theory - A *probabilistic experiment* is an experiment whose outcome is not known before its realization despite the set of all possible outcomes being known in advance. This set is called the *sample space* S of the probabilistic experiment. (16)

Any subset E of the sample space is called an *event*. A probability function $p(E)$ is a function that maps events into real numbers and satisfy the Kolmogorov axioms:

(a) $0 \leq p(E) \leq 1$;

(b) $p(S) = 1$;

(c) $p(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} p(E_i)$, if $E_i \cap E_j = \emptyset$ for $i \neq j$.

Let $p(E, F) := p(E \cap F)$ be the *joint probability* of events E and F, i.e., the probability that both events occur in a probabilistic experiment. Then, the probability that event E occurs given that event F has already occurred is a *conditional probability*, which is defined as

$$p(E|F) := \frac{p(E, F)}{p(F)}, \text{ for } p(F) > 0. \tag{2.1}$$

In most probabilistic experiments we are interested in the calculation of some function of the outcomes, which might be a difficult task considering the outcomes are subsets of the sample space and not numbers. A function $X(E)$ that associates outcomes of the experiment (events) with real numbers is called a *random variable*.

Stochastic processes - When studying some probabilistic experiment that evolves on time, the adequate approach is to use a different random variable for each instant of time. A *stochastic process* is a family $\{X_t\}_t$ of random variables, in which X_t is the random variable associated to the probabilistic experiment at time t, and the time variable t may belong to some (discrete) set $\{t_0, t_1, ..., t_n\}$ or some (continuous) interval $[t_0, t_f]$. (17) From now on, for sake of simplicity and agreement with the quantum case, we only consider discrete-time evolutions.

In a stochastic process, at each instant of time t_s the possible outcomes are denoted by x_s. We shall refer to the *probability vector* $\boldsymbol{p}(x_s,t_s) := (p(x_1,t_s), p(x_2,t_s), ..., p(x_m,t_s))$ as describing the *state* of the stochastic process at some instant t_s[1].

The joint probability $p(x_n,t_n;x_{n-1},t_{n-1};...;x_1,t_1;x_0,t_0)$ provides the probability that the events x_s occur at times t_s. The stochastic process is fully described by a complete family of joint probabilities, that is, the probabilities of all possible combinations of outcomes happening at each instant of time.

Considering that any outcome might only be affected by the outcomes that happened before it and might only affect the outcomes that will happen after it, the natural way to define a conditional joint probability for a stochastic process is

$$p(x_n,t_n;...;x_{s+1},t_{s+1}|x_s,t_s;...;x_0,t_0) := \frac{p(x_n,t_n;...;x_0,t_0)}{p(x_s,t_s;...;x_0,t_0)}, \text{ for } p(x_s,t_s;...;x_0,t_0) > 0. \tag{2.2}$$

It follows naturally from this definition that the probability of obtaining any outcome x_s at some time t_s may be written in terms of the probabilities of the possible outcomes x_r in some prior time t_r

$$p(x_s,t_s) = \sum_{x_r} p(x_s,t_s|x_r,t_r)p(x_r,t_r), \tag{2.3}$$

or in matrix form, writing the probability vector as a column matrix,

$$\begin{bmatrix} p(x_1,t_s) \\ p(x_2,t_s) \\ . \\ . \\ . \end{bmatrix} = T(t_s,t_r) \begin{bmatrix} p(x_1,t_r) \\ p(x_2,t_r) \\ . \\ . \\ . \end{bmatrix}, \tag{2.4}$$

where $T(t_s,t_r)$ is the *transition matrix* (from time t_r to time t_s), whose elements are defined as $T_{ij} := p(x_i,t_s|x_j,t_r)$, from which follows that it is a *stochastic matrix*[2].

Divisibility - A stochastic process is called divisible if the intermediate transition matrices $T(t_s,t_r)$ satisfy the Chapman-Kolmogorov condition

$$T(t_s,t_0) = T(t_s,t_r)T(t_r,t_0), \tag{2.5}$$

for all $0 < r < s \leq n$ [3].

[1] It is actually far more common in the literature to refer to the outcome that occurred at some instant as being the state of the process at that instant. We choose to call the probability vector the state of the system for an easier later connection with the quantum case.

[2] Stochastic matrices are those that satisfy the positivity and normalization conditions: $T_{ij} \geq 0,\ \sum_i T_{ij} = 1,\ \forall i,j$.

[3] If you search for the word "divisibility" in any stochastic processes textbook you will only find the concept of "Infinite

When we ask if a certain stochastic process is divisible what we want to know is: Is it possible to obtain the correlation between any two given non-consecutive instants of time if we know the correlation between each of the instants and some intermediate time?[4] If the answer is yes, then the process is divisible. If the answer is no, then the process is not divisible.

Notice that if we have the initial state $\boldsymbol{p}(x_0, t_0)$ of the stochastic process and the complete family of transition matrices $\{T(t_s, t_r)\}_{s>r\geq 0}^{n}$ we could obtain the *bipartite marginal probabilities* $\{p(x_s, t_s; x_r, t_r)\}_{s>r\geq 0}^{n}$ relating every pair of instants of time of the stochastic processes. This means that a description of the process based on the transition matrices is equivalent to the one based on the set $\{p(x_s, t_s; x_r, t_r)\}_{s>r\geq 0}^{n}$.

Considering that the Chapman-Kolmogorov condition might be written just in terms of the bipartite marginals as

$$p(x_s, t_s; x_q, t_q) = \sum_{x_r} \frac{p(x_s, t_s; x_r, t_r)p(x_r, t_r; x_q, t_q)}{p(x_r, t_r)}, \text{ for all } 0 \leq q < r < s \leq n, \tag{2.6}$$

it follows that it is possible to verify if the process is divisible even in the case in which we only know the bipartite marginal distributions. This fact together with the intuition we gave on the correlations of a divisible process provide us with an important remark on divisibility: divisibility is a property related to the **two-time** correlations of a given stochastic process.

It is important to notice that the bipartite marginals can be obtained from the complete family of joint probabilities, but cannot be used to reconstruct them.

Markovianity - A stochastic process is called *Markovian* if, for the whole family of joint probabilities describing it, it is true that

$$p(x_{s+1}, t_{s+1}|x_s, t_s; x_{s-1}, t_{s-1}; ...; x_0, t_0) = p(x_{s+1}, t_{s+1}|x_s, t_s), \text{ for all } 0 < s < n. \tag{2.7}$$

One way to phrase this property is to say that in a Markovian process the state of the system at the future time t_{s+1} depends only on the state of the system at the present time t_s and not on any past

divisibility", which has nothing to do with what we are defining here. It seems like the definition of divisibility of stochastic processes was imported by physicists from quantum theory to probability theory in Ref. (8) in order to study the relation between divisibility and Markovianity. However, the definition presented there is not clear in a few aspects: first, the authors define the transition matrix as the one that connects the probability vectors from two instants of time, but they never mention the fact that such a matrix is not uniquely defined; second, the authors state that the elements of the intermediate transition matrices are not always the conditional probabilities, but by comparing Eq. (2.4) with the *Law of Total Probability* $P(A) = \sum_n P(A|B_n)P(B_n)$ one can conclude that the matrix of conditional probabilities is always a transition matrix of any two given probability vectors; third, the authors define divisible processes as those whose intermediate transition matrices are stochastic and satisfy the Chapman-Kolmogorv condition, but again it is not mentioned which of the possible transition matrices they are referring to. Based on Ref. (15), we avoid all these issues by defining the transition matrices exactly as those whose elements are the conditional probabilities. Since this guarantees that they are stochastic matrices, our definition of divisibility relies only on the Chapman-Kolmogorov condition.

[4]By "correlation" here we do not mean the well defined statistical coefficient (16), but just some relation existing between the outcomes of the two random variables.

state. Therefore, since they do not carry any information about past states, Markovian processes are also called *memoryless* processes.

It is straightforward to show that any Markovian process is also divisible, since it provides a direct way to satisfy the Chapman-Kolmogorov condition. There is a natural intuition to this result: in a Markovian process the correlation between any two non-consecutive instants of time is just the consequence of the correlations between the consecutive instants of time that connect them, so the process has to be divisible.

It is also clear that we must check the whole family of joint probabilities in order to determine if a stochastic process is Markovian or not. This means that despite the bipartite marginals being sufficient for us to verify if a process is divisible, they are not sufficient for us to verify if the process is Markovian. This reveals an important aspect of Markovianity: Markovianity is a property related to the **multi-time** correlations of a given stochastic process.

Examples - We shall present some examples of stochastic processes to illustrate the aforementioned properties. In order to keep things simple, these processes have three instants of time and only two possible events at each time, which are denoted by 0 or 1.

1) Consider a process where the outcome 1 either happens in only one of the instants or in all three of them, with the following probability distribution (8)

x_0	x_1	x_2	p
1	0	0	$1/4$
0	1	0	$1/4$
0	0	1	$1/4$
1	1	1	$1/4$

The bipartite marginals are given by $p(x_i = a, t_i; x_j = b, t_j) = 1/4$ for $i, j = 0, 1, 2$, with $i \neq j$ and $a, b = 0, 1$. It is easy to verify that these marginals satisfy Eq. (2.6), but the joint probability does not satisfy Eq. (2.7), so the process is divisible but non-Markovian.

2) Consider a process in which all the random variables are independent and identically distributed, so that the family of joint probabilities is given by

$$p(x_0 = a, t_0; x_1 = b, t_1; x_2 = c, t_2) = 1/8, \text{ for all } a, b, c = 0, 1. \tag{2.8}$$

The bipartite marginals for this process are exactly the same as those of the last example, which means this process is also divisible. The main difference is that the joint probability in this case satisfies Eq. (2.7), so this is actually a Markovian stochastic process, as we would expect.

3) Consider a stochastic process in which the outcomes of instants t_0 and t_2 are always the same, but they are independent of the outcome of instant t_1, such that the joint probability has the following distribution.

x_0	x_1	x_2	p
0	0	0	0.1
0	1	0	0.2
1	0	1	0.3
1	1	1	0.4

The marginals in this case are given by

x_0 or x_2	x_1	p
0	0	0.1
0	1	0.2
1	0	0.3
1	1	0.4

and $p(x_0 = 0, t_0; x_2 = 0, t_2) = 0.4$, $p(x_0 = 1, t_0; x_2 = 1, t_2) = 0.6$. We can show that these marginals do not satisfy Eq. (2.6), so this stochastic process is not divisible. As we could already see from the beginning, since x_0 and x_2 are clearly correlated, this process is not Markovian.

Description of the process - As it was also shown in the examples above, the family of bipartite marginals provides an incomplete description of the stochastic process. The processes from examples 1) and 2) have the same marginal distributions, but one of them is Markovian and the other one is not. When we average the joint probabilities over time steps in order to obtain the bipartite marginals we lose information about the multi-time correlations, which would be necessary for us to verify if the process is Markovian and we are left just with a family of two-time correlations.

3 Quantum Dynamical Mappings

In this chapter we present the formal structure of quantum theory and then, by noticing its similarities with stochastic processes theory, we construct the framework of quantum dynamical mappings. We also discuss why it is not possible to naturally extend the concept of Markovianity from stochastic processes to quantum dynamical mappings.

Quantum theory - Any physical system S is associated to a *Hilbert space* $\mathcal{H}_S$, called the *state space* of the system. Let $L(\mathcal{H}_S)$ be the space of all linear operators acting on $\mathcal{H}_S$. The *state* of the system is fully described by a *density operator* $\rho \in L(\mathcal{H}_S)$, i.e., a linear operator that is also *positive semidefinite* and has *unitary trace*[5]. Given another system E, whose state space is $\mathcal{H}_E$, the state space of the *composite system* SE is the tensor product $\mathcal{H}_S \otimes \mathcal{H}_E$ of the state spaces of its subsystems. (18)

A *measurement* is described by a set $\{M_m\}$ of *measurement operators*. The probability of obtaining the outcome m when measuring the state ρ is $p(m) = \mathrm{Tr}\left[M_m^\dagger M_m \rho\right]$ and the state of the system after obtaining this outcome is $\rho_m = M_m \rho M_m^\dagger / p(m)$.

The dynamics of any *closed quantum system* is generated by a *Hermitian* operator H, which is called the *Hamiltonian* of the system. Given the Hamiltonian H and the initial state $\rho(t_0)$, we can solve the *Schrödinger equation*

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] \tag{3.1}$$

and obtain the state of the system at any time.

The statement above should be taken as the definition of a closed quantum system. It is wrong to state that closed systems are those that do not interact with any other system. If the evolution of the system due to this interaction can be described by some Hamiltonian, it still is a closed system. Any quantum system that is not closed is called an open quantum system.

Open system dynamics - The evolution of open quantum systems is not mentioned on the postulates that define quantum theory as it can be derived from them. Given our system of interest S, if we consider that it is coupled to another system E, its *environment*, such that the composite system is closed[6], then the global dynamics can be calculated via the Schrödinger equation and we obtain the dynamics for S just by discarding the environment degrees of freedom[7]. This is what we shall call

[5] A positive semidefinite operator is one whose eigenvalues are real and non-negative. The unitary trace condition is simply $\mathrm{Tr}[\rho] = 1$.

[6] The fact that we can always find an environment such that the composite system is closed is a consequence of the *Stinespring-Kraus theorem*. (19) We shall discuss more about this theorem in the next chapter.

[7] The mathematical tool to do this is the partial trace. (18)

the *subsystem dynamics* approach.

Although this approach provides us an exact solution for the evolution of some systems and an approximate one for most of them, it is not so useful when we want to study large classes of physical systems. Finding the evolution of every single system individually and then trying to investigate general properties is an exhausting task in most cases. Therefore, we should seek a framework that generally describes the evolution of open quantum systems without specifying the details of the physical system we are working with. (20)

In order to achieve this goal, we notice that the way quantum theory and stochastic processes theory describe states and evolutions are intimately related. Thus, we shall borrow some tools from the latter formalism, like the evolution via linear operators and the divisibility property, and use them in the former, as depicted in the next sections.

States - Since the density operator ρ_S describing the state of a quantum system is positive semidefinite, it has a spectral decomposition

$$\rho_S = \sum_i p(x_i) \left|\psi_i\right\rangle \left\langle\psi_i\right| , \tag{3.2}$$

where $\left|\psi_i\right\rangle \left\langle\psi_i\right|$ are projectors into its eigenspaces and $p(x_i)$ are its eigenvalues, which can be regarded as probabilities due to the positivity of ρ_S. Therefore, the state of any quantum system can be uniquely associated to some probability vector $\boldsymbol{p}(x_i) = (p(x_1), p(x_2), ..., p(x_n))$, which as well describes the state of some suitable stochastic process. This is the fundamental bridge between the two theories.

Evolution - For any stochastic process, the state of the process at any time t might be obtained by means of a transition matrix $T(t, t_0)$ acting on the initial state $\boldsymbol{p}(x_0, t_0)$. Analogously, the state of any quantum system at some time t is related to the initial state $\rho_S(t_0)$ by some *quantum channel* $\mathcal{E}(t, t_0)$.

Since quantum states are described by density operators, we could use *super operators* that map density operators into density operators to describe the evolution of quantum systems. These super operators are what we call quantum channels. (19) They must have the following properties:

- linearity, ensuring they map linear operators into linear operators;
- trace preservation, in order to map unitary traced operators into unitary traced operators;
- complete positivity, so they map positive semidefinite operators into positive semidefinite operators.

To be more precise, super operators that map positive operators into positive operators are called just *positive*. However, if we a have a density operator of a composite system and apply a positive super operator only in a subsystem and do nothing on the rest, the resulting operator describing the state of the subsystem might not be a positive one, which does not make physical sense. In order to be sure that such a thing will not happen, we require the tensor product of the quantum channel and the identity super operator of any other Hilbert space to be a positive super operator. This is what we call complete positivity.

Linearity and trace preservation are rather straightforward to be secured. Complete positivity, on the other hand, requires that we severely restrict the set of possible initial system-environment states, so that the evolution is consistent with the subsystem dynamics approach. (21)

The fact that most states in which the system of interest is entangled with its environment cannot be mapped into positive operators by means of a super operator is a well known problem of the quantum channel formalism that we shall not discuss here[8] Instead, we will avoid this issue by considering that our initial system-environment state $\rho_{SE}(t_0)$ is always a product state, such that there will always exist a quantum channel $\mathcal{E}(t, t_0)$ mapping the initial state $\rho_S(t_0)$ of the system to any later time t

$$\rho_S(t) = \mathcal{E}(t, t_0)\rho_S(t_0). \tag{3.3}$$

Since non-trivial dynamics usually entangle the system and the environment, in general the intermediate quantum channels $\mathcal{E}(t, t')$, for $t_0 < t' < t$, are not well defined.

Quantum dynamical mappings - The discrete-time evolution of any quantum system can be described by a complete family of quantum channels $\mathcal{M} = \{\mathcal{E}(t_s, t_0)\}_{s>0}^n$, which is called a *quantum dynamical mapping*. This is a widely used approach in the quantum information field due to its generality in describing the evolution of open quantum systems.

Divisibility - A quantum dynamical mapping $\mathcal{M} = \{\mathcal{E}(t_n, t_0)\}_{s>0}^n$ is called divisible if the intermediate super operators $\mathcal{E}(t_s, t_r)$ are well defined quantum channels and satisfy the Chapman-Kolmogorov condition

$$\mathcal{E}(t_s, t_0) = \mathcal{E}(t_s, t_r)\mathcal{E}(t_r, t_0). \tag{3.4}$$

Markovianity - Given that the quantum dynamical mappings formalism is widely used in quantum information theory, having a good definition of a Markovian quantum dynamical mapping would be of great importance to the field, since it would indicate which physical processes keep information from past states and which are memoryless.

However, a problem emerges at this point: there is a fundamental difference on how measurements

[8]For more on this topic see, for example, Ref. (21)

occur in each theory. In the stochastic processes formalism a measurement result is just an element taken from a set of possible results (the sample space). In quantum theory, on the other hand, there are infinitely many possible measurements that can be performed in any given quantum system. This means that the measurement result does not only depend on which element was taken from the set of possible results, but also on which measurement was performed.

Since the choice of the measurement depends on an external agent, an *experimenter*, which is not described by the theory, it is apparently not possible to define a quantum analog of conditional probabilities. Without such a definition there is no natural way to extend the concept of Markovianity from stochastic processes to quantum dynamical mappings.

The best thing that can be done is proposing a definition for quantum Markovianity that resembles some aspects of the classical case. In the literature there are many non-equivalent definitions in this sense, some of which we shall present and discuss in the following chapter.

4 Quantum Markovianity

In this chapter we present some of the main definitions of quantum Markovianity proposed in the literature and discuss why it is not possible to properly define Markovianity for quantum dynamical mappings. In the end we briefly introduce an alternative framework in which such a definition is made possible.

RHP definition - Rivas, Huelga and Plenio suggested that Markovianity should be taken as equivalent to divisibility. (8) The reason for this comes from the *Stinespring-Kraus theorem*. This theorem states that the evolution given by any quantum channel is equivalent to the evolution obtained by a subsystem dynamics approach for some suitable finite-dimensional environment and global Hamiltonian. It follows that, for some experimenter with access only to the system, any divisible dynamics is completely indistinguishable from a dynamics in which the system and the environment interact at each time step and then the environment is discarded. Therefore, the environment cannot keep information about past states of the system, which means that there should be no memory effects during the dynamics.

BLP definition - Breuer, Laine and Piilo proposed a definition of Markovianity based on an information quantifier. (9, 22) The key idea here is that a measure of distinguishability can be taken as a good measure of information, since the more information an experimenter has about a system, the easier it is for it to distinguish any two given states of that system.

Their measure of distinguishability is constructed as follows. Suppose Alice prepares a quantum system in one of two possible states ρ_1 and ρ_2 with equal probability and give it to Bob. Bob can perform a measurement in the system he received and guess which one was. The probability that Bob guesses correctly is a linear function of $D(\rho_1, \rho_2)$, the *trace distance* between the two states.

With this in mind, we can postulate that a process is Markovian if and only if the trace distance between any pair of states is monotonically non-increasing along the dynamics. This means that the amount of information the experimenter has about the system only decreases with time in a Markovian process.

It has been shown that this definition of Markovianity is weaker than divisibility. This means that, according to the BLP definition, divisibility would imply Markovianity, but the converse might not hold.

Guessing probability - Buscemi and Datta generalized the aforementioned information quantifier by considering the *guessing probability*, which is the probability of distinguishing elements of an ensemble $\{\rho_x; p(x)\}$[9]. (23) In the particular case that the ensemble is constituted only by a pair of equiprobable states we return to the trace distance measure.

It was shown that the guessing probability is monotonically non-increasing along some dynamics if and only if the dynamics is divisible. This means that if we define Markovian processes as those in which the guessing probability is monotonically non-increasing, we are at the same time generalizing the BLP definition and recovering the RHP definition.

The last result indicates that any non-divisible dynamics will necessarily show memory effects that can be detected by means of the guessing probability. On the other hand, as we already discussed, the Stinespring-Kraus theorem provides a strong argument on why there should be no memory effects in a divisible dynamics. Therefore, it may look like we have reached an end to the discussion: quantum Markovianity should be taken as equivalent to divisibility.

Non-correspondence between the formalisms - There is, however, one aspect of this last definition that can be considered an issue. If we are assuming that the stochastic processes formalism is a particular case of quantum dynamical mappings, namely, the one obtained when we fix one measurement for each time step, we should expect all divisible stochastic processes to be Markovian, as it is the case for quantum dynamical mappings. However, as already discussed, there are divisible stochastic processes that are non-Markovian.

After a thorough analysis of the structure of both formalisms, it turns out that the problem was in the first assumption we made: stochastic processes are not a particular case of quantum dynamical mappings. Besides all the aforementioned similarities between the two theories, quantum dynamical mappings way of describing the process is somewhat simpler than that of stochastic processes.

If we fixed one measurement for each time step of some quantum dynamical mapping we would obtain a description of the dynamics of the quantum system that is equivalent to that of transition matrices for stochastic processes. (15) As we mentioned in Chapter 2, this is the same as a description based on bipartite marginal distributions, which means that quantum dynamical mappings describe processes by means of a set of two-time correlations. As we discussed back there, it is possible to determine if a process is divisible just with its marginal distributions, therefore it makes sense that the definition of divisibility for quantum dynamical mappings can be directly extended from stochastic processes.

[9]An ensemble $\{\rho_x; p(x)\}$ is simply a set of pairs constituted by a state ρ_x and a probability p_x of that state occurring.

On the other hand, we also showed that it is not possible to determine if a process is Markovian just by checking its bipartite marginals as we must have the joint probabilities to do so. That is the reason why we cannot properly define Markovianity for quantum dynamical mappings. Since Markovianity is a property on the multi-time correlations of the process and quantum dynamical mappings only carry two-time correlations, it does not make much sense to say if a quantum dynamical mapping is Markovian or not.

If we insist on having such a definition, we could say that Markovianity of quantum dynamical mappings should address the only multi-time correlations that exist in its description, i.e., the two-time correlations. In this case, Markovianity should simply be equivalent to divisibility, which is the conclusion at which we had arrived by examining the definitions proposed in the literature.

Notice that none of this means that there are no multi-time correlations in quantum processes, it just means that we lose information about such correlations when we use quantum dynamical mappings to describe the process, the same thing that happens in stochastic processes when we average joint probabilities over time steps to obtain the bipartite marginals.

Therefore, if we had a framework that generalized the joint probabilities description to the quantum case, where there are many possible measurements at each time step, these multi-time correlations would not be lost and we would be able to rigorously define quantum Markovianity.

Process tensor - It turns out that such a description of quantum processes already exists and is known as the *process tensor* framework. The process tensor is the mathematical object that carries all the information about the unitary evolution of the global (system plus environment) state between the instants of time. It also provides the initial global state.

At each time, the system might undergo a *control operation* that is performed by an external agent. This control operation is anything that can be represented by a completely positive linear operator, like a deterministic evolution, a measurement, a preparation of a new state, or some combination of those. The information of which control operations are performed at each instant of time is given in another tensor, which we could call the *operations tensor*.

This way, we perfectly split the contributions of the uncontrolled environment (process tensor) from the contributions of the controlled operations (operations tensor). The final state of the system is obtained by simply contracting the tensors[10].

[10] We have just presented the basic ideas and motivations behind the process tensor framework. For a rigorous mathematical definition see, for example, Refs. (24, 25)

With this approach there are no issues with initial system-environment correlations. For any initial global state and any global dynamics the result will be a well defined density operator describing the final state of the system.

Since the process tensor carries multi-time correlations it is possible to have a good definition of quantum Markovianity. First of all, suppose that at some point of the process we discard the state of the system and prepare a fresh new one, independent of the one we just discarded, and put on its place. This is what we call a *causal break*. Now, we define: a quantum process is Markovian if, after any causal break and for any sequence of control operations, the final state of the system is independent of the control operations performed before the causal break.

If we choose each control operation to be a fixed measurement, then we are back to stochastic processes. It can be shown that in this case the process tensor is equivalent to a complete family of joint probabilities and that our definition of Markovianity reduces to the one we already knew for stochastic processes. (24)

Therefore, we could say that the process tensor solves the two main problems of quantum dynamical mappings: the initial correlations restriction and the definition of Markovianity. For this reason the process tensor framework might be considered far more adequate than quantum dynamical mappings for most informational analysis of quantum systems[11].

[11] Due to its extremely recent formalization, there are not yet many works in the literature that use the process tensor framework. A few of them can be found in Refs. (25, 26, 27)

5 Conclusions

In this undergraduate thesis we introduced both stochastic processes and quantum theory formalisms and discussed the importance of obtaining a general description for the evolution of open quantum systems. We then extended some concepts of stochastic processes to quantum theory and constructed the framework of quantum dynamical mappings, which is a powerful and widely used tool for describing the dynamics of open quantum systems.

It was also shown that, despite the natural generalizations of the transition matrix and the divisibility property to the quantum case, it is not possible to directly extend the concept of Markovianity from stochastic processes to quantum dynamical mappings. Such an extension would be key for the analysis of information flow in quantum systems and could have a deep impact in our understanding of the behavior of open systems.

We thus presented some of the main proposals of definitions of quantum Markovianity that can be found in the literature and explained how they are all inconsistent with the definition of Markovianity for stochastic processes. Then, it was discussed that the problem of defining Markovianity for quantum dynamical mappings has its roots on the structure of the formalism: its description is based on a family of two-time correlations while Markovianity is about multi-time correlations.

Finally, we briefly presented the process tensor framework, which is a rather new approach to the description of open quantum systems dynamics. Since the process tensor carries multi-time correlations, it enables us to define quantum Markovianity in a natural way that is a consistent generalization of the stochastic processes case. Therefore, this framework presents itself as a more adequate one for studying quantum systems from an informational point of view.

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